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# Large Generalized Books are $p$-Good

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**Abstract**

Let $B_q^{(r)} = K_r + qK_1$ be the graph consisting of $q$ distinct $(r+1)$-cliques sharing a common $r$-clique. We prove that if $p \geq 2$ and $r \geq 3$ are fixed, then

$$r(K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$$

for all sufficiently large $q$.

Keywords: Ramsey numbers; p-good; generalized books; Szemerédi lemma

## 1 Introduction

The title of this paper refers to the notion of goodness introduced by Burr and Erdős in [3] and subsequently studied by Burr and various collaborators. A connected graph $H$ is $p$-good if the Ramsey number $r(K_p, H)$ is given by

$$r(K_p, H) = (p-1)(|V(H)| - 1) + 1.$$

In this paper we prove that for every $p \geq 3$ the generalized book $B_q^{(r)} = K_r + qK_1$ is $p$-good if $q$ is sufficiently large.

As much as possible, standard notation is used; see, for example, [2]. A set of cardinality $p$ is called a $p$-set. Unless explicitly stated, all graphs are defined on the vertex set $[n] = \{1, 2, \ldots, n\}$. Let $u$ be any vertex; then $N_G(u)$ and $d_G(u) = |N_G(u)|$ denote its neighborhood

and degree respectively. A graph with $n$ vertices and $m$ edges will be designated by $G(n,m)$. By an $r$-*book* we shall mean some number of independent vertices that are each connected to every vertex of an $r$-clique. The given $r$-clique is called the *base* of the $r$-book and the additional vertices are called the *pages*. The number of pages of an $r$-book is called its *size*; the size of the largest $r$-book in a graph $G$ is denoted by $bs^{(r)}(G)$. We shall denote the complete $p$-partite graph with each part having $q$ vertices by $K_p(q)$. The *Ramsey number* $r(H_1,H_2)$ is the least number $n$ such that for every graph $G$ of order $n$ either $H_1 \subset G$ or $H_2 \subset \overline{G}$.

## 2 The structure of subsaturated $K_{p+1}$-free graphs

We shall need the following theorem of Andrásfai, Erdős and Sós [1].

**Theorem 1** *If $G$ is a $K_{p+1}$-free graph of order $n$ and*

$$\delta(G) > \left(1 - \frac{3}{3p-1}\right) n,$$

*then $G$ is $p$-chromatic.* $\square$

The celebrated theorem of Turán gives a tight bound on the maximum size of a $K_p$-free graph of given order. In the following theorem we show that if the size of a $K_{p+1}$-free graph is close to the maximum then we may delete a small portion of its vertices so that the remaining graph is $p$-chromatic. This is a particular stability theorem in extremal graph theory (see [9]).

**Theorem 2** *For every $p \geq 2$ there exists $c = c(p) > 0$, such that for every $\alpha$ satisfying $0 < \alpha \leq c$, every $K_{p+1}$-free graph $G = G(n,m)$ satisfying*

$$m \geq \left(\frac{p-1}{2p} - \alpha\right) n^2$$

*contains an induced $p$-chromatic graph $G_0$ of order at least $\left(1 - 2\alpha^{1/3}\right) n$ and with minimum degree*

$$\delta(G_0) \geq \left(1 - \frac{1}{p} - 4\alpha^{1/3}\right) n.$$

**Proof** Let $c_0$ be the smallest positive root of the equation

$$x^3 + \left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right) x - \frac{1}{2\,(3p-1)\,p} = 0 \tag{1}$$

and set $c\,(p) = c_0^3$; then, for every $y$ satisfying $0 < y \leq c\,(p)\,,$ we easily see that

$$y + \left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right) y^{1/3} \leq \frac{1}{2\,(3p-1)\,p}. \tag{2}$$

A rough approximation of the function $c\,(p)$ is $c(p) \approx 6^{-3}p^{-6}$, obtained by neglecting the $x^3$ term in equation (1) and substituting the appropriate asymptotic (for large $p$) approximations for the remaining coefficients. This gives reasonable values even for small $p$. For all $p \geq 2$,

$$\frac{1}{(2p(3p+2))^3} < c(p) < \frac{1}{(2p(3p-1))^3}. \tag{3}$$

The upper bound is evident, and the lower bound follows from a simple computation.

Let $0 < \alpha \leq c\,(p)$ and the graph $G = G\,(n, m)$ satisfy the hypothesis of the theorem. We shall prove first that

$$\sum_{u=1}^{n} d^2\,(u) \leq 2\left(\frac{p-1}{p}\right) mn. \tag{4}$$

Indeed, writing $k_3(G)$ for the number of triangles in $G$, we have

$$3k_3(G) = \sum_{uv \in E} |N(u) \cap N(v)| \geq \sum_{uv \in E} (d(u) + d(v) - n) = \sum_{u=1}^{n} d^2(u) - mn.$$

Applying Turán's theorem to the $K_p$-free neighborhoods of vertices of $G$, we deduce

$$3k_3(G) \leq \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u).$$

Hence,

$$\sum_{u=1}^{n} d^2(u) - mn \leq \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u)$$

and (4) follows.

Since $0 < \alpha \leq c(p)$, taking the upper bound in (3) for $p = 2$, we see that $\alpha \leq 20^{-3}$. Hence,

$$\begin{aligned}(1+8\alpha)\frac{4m^2}{n} &\geq 2(1+8\alpha)\left(\frac{p-1}{p}-2\alpha\right)mn \\ &= 2\left(\frac{p-1}{p}+\left(6-\frac{8}{p}\right)\alpha-16\alpha^2\right)mn \\ &\geq 2\left(\frac{p-1}{p}+2\alpha-16\alpha^2\right)mn > 2\left(\frac{p-1}{p}\right)mn,\end{aligned}$$

and from (4) we deduce

$$\begin{aligned}\sum_{u=1}^{n}\left(d(u)-\frac{2m}{n}\right)^2 &= \sum_{u=1}^{n}d^2(u)-\frac{4m^2}{n} \leq 2\left(\frac{p-1}{p}\right)mn-\frac{4m^2}{n} \\ &< 8\alpha\frac{4m^2}{n} \leq 8\alpha\left(\frac{p-1}{p}\right)^2 n^3. \qquad (5)\end{aligned}$$

Set $V = V(G)$ and let $M_\varepsilon$ be the set of all vertices $u \in V$ satisfying $d(u) < 2m/n - \varepsilon n$. For every $\varepsilon > 0$, inequality (5) implies

$$|M_\varepsilon|\,\varepsilon^2 n^2 < \sum_{u\in M_\varepsilon}\left(d(u)-\frac{2m}{n}\right)^2 \leq \sum_{u\in V}\left(d(u)-\frac{2m}{n}\right)^2 \leq 8\alpha\left(\frac{p-1}{p}\right)^2 n^3,$$

and thus,

$$|M_\varepsilon| < 8\varepsilon^{-2}\alpha\left(\frac{p-1}{p}\right)^2 n. \qquad (6)$$

Furthermore, setting $G_\varepsilon = G[V\backslash M_\varepsilon]$, for every $u \in V(G_\varepsilon)$, we obtain

$$d_{G_\varepsilon}(u) \geq d(u) - |M_\varepsilon| \geq \frac{2m}{n} - \varepsilon n - |M_\varepsilon| > \frac{p-1}{p}n - 2\alpha n - \varepsilon n - |M_\varepsilon|. \qquad (7)$$

For $\varepsilon = 2\alpha^{1/3}$ we claim that

$$\frac{p-1}{p}n - 2\alpha n - \varepsilon n - |M_\varepsilon| > \frac{3p-4}{3p-1}(n - |M_\varepsilon|) = \frac{3p-4}{3p-1}v(G_\varepsilon). \qquad (8)$$

Indeed, assuming the opposite and applying inequality (6) with $\varepsilon = 2\alpha^{1/3}$, we see that

$$\left(\frac{1}{(3p-1)p} - 2\alpha - 2\alpha^{1/3}\right)n \leq \frac{3}{3p-1}|M_{2\alpha^{1/3}}| < 2\frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\alpha^{1/3}n;$$

hence,

$$2\alpha + 2\left(1 + \frac{3}{3p-1}\left(\frac{p-1}{p}\right)^2\right)\alpha^{1/3} - \frac{1}{(3p-1)\,p} > 0,$$

contradicting (2).

Set $G_0 = G_{2\alpha^{1/3}}$; from (8), we see that $G_0$ satisfies the conditions of Theorem 1, so it is $p$-chromatic.

Finally, from (6) and (7), we have

$$\delta\left(G_0\right) \geq \frac{p-1}{p}n - 2\alpha n - 2\alpha^{1/3}n - \left(\frac{p-1}{p}\right)^2\alpha^{1/3}n > \frac{p-1}{p}n - 2\alpha n - 3\alpha^{1/3}n$$

$$> \left(1 - \frac{1}{p} - 4\alpha^{1/3}\right)n,$$

completing the proof. $\square$

# 3 A Ramsey property of $K_{p+1}$-free graphs

The main result of this section is the following theorem.

**Theorem 3** *Let $r \geq 2$, $p \geq 2$ be fixed. For every $\xi > 0$ there exists an $n_0 = n_0(p, r, \xi)$ such that every graph $G$ of order $n \geq n_0$ that is $K_{p+1}$-free either satisfies $bs^{(r)}\left(\overline{G}\right) > n/p$, or contains an induced $p$-chromatic graph $G_1$ of order $(1-\xi)\,n$ and minimum degree*

$$\delta(G_1) \geq \left(1 - \frac{1}{p} - 2\xi\right)n.$$

Our main tool in the proof of Theorem 3 is the regularity lemma of Szemerédi (SRL for short); for expository matter on SRL see [2] and [7]. For the sake of completeness we formulate here the relevant basic notions.

Let $G$ be a graph; if $A, B \subset V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of $A - B$ edges and call the value

$$d(A, B) = \frac{e(A, B)}{|A|\,|B|}$$

the *density* of the pair $(A, B)$.

Let $\varepsilon > 0$; a pair $(A, B)$ of two nonempty disjoint sets $A, B \subset V(G)$ is called *$\varepsilon$-regular* if the inequality

$$|d(A, B) - d(X, Y)| < \varepsilon$$

holds whenever $X \subset A$, $Y \subset B$, $|X| \geq \varepsilon |A|$, and $|Y| \geq \varepsilon |B|$.

We shall use SRL in the following form.

**Theorem 4 (Szemerédi's Regularity Lemma)** *Let $l \geq 1$, $\varepsilon > 0$. There exists $M = M(\varepsilon, l)$ such that, for every graph $G$ of sufficiently large order $n$, there exists a partition $V(G) = \cup_{i=0}^{k} V_i$ satisfying $l \leq k \leq M$ and:*

(i) $|V_0| < \varepsilon n$, $|V_1| = ... = |V_k|$;

(ii) *all but at most $\varepsilon k^2$ pairs $(V_i, V_j)$, $(i, j \in [k])$, are $\varepsilon$-uniform.*

We also need a few technical results; the first one is a basic property of $\varepsilon$-regular pairs (see [7], Fact 1.4).

**Lemma 1** *Suppose $0 < \varepsilon < d \leq 1$ and $(A, B)$ is an $\varepsilon$-regular pair with $e(A, B) = d|A||B|$. If $Y \subset B$ and $(d - \varepsilon)^{r-1}|Y| > \varepsilon|B|$ where $r > 1$, then there are at most $\varepsilon r|A|^r$ $r$-sets $R \subset A$ with*

$$\left| \left( \bigcap_{u \in R} N(u) \right) \cap Y \right| \leq (d - \varepsilon)^r |Y|.$$

The next lemma gives a lower bound on the number of $r$-cliques in a graph consisting of several dense $\varepsilon$-regular pairs sharing a common part.

**Lemma 2** *Suppose $0 < \varepsilon < d \leq 1$ and $(d - \varepsilon)^{r-2} > \varepsilon$. Suppose $H$ is a graph and $V(H) = A \cup B_1 \cup \cdots \cup B_t$ is a partition with $|A| = |B_1| = \cdots = |B_t|$ and such that for every $i \in [t]$ the pair $(A, B_i)$ is $\varepsilon$-regular with $e(A, B_i) \geq d|A||B_i|$. If $m$ is the number of the $r$-cliques in $A$, then at least*

$$t|A|(m - \varepsilon r|A|^r)(d - \varepsilon)^r$$

*$(r+1)$-cliques of $H$ have exactly $r$ vertices in $A$.*

**Proof** Set $a = |A| = |B_1| = \cdots = |B_t|$. For every $i \in [t]$, applying Lemma 1 to the pair $(A, B_i)$ with $Y = B_i$ we conclude that there are at most $\varepsilon r a^{r-1}$ $r$-sets $R \subset A$ with

$$\left| \left( \bigcap_{u \in R} N(u) \right) \cap B_i \right| \leq (d - \varepsilon)^r a,$$

and therefore, at least $(m - \varepsilon r a^r)$ $r$-cliques $R \subset A$ satisfy

$$\left| \left( \bigcap_{u \in R} N(u) \right) \cap B_i \right| > (d - \varepsilon)^r a.$$

Hence, at least $t(d - \varepsilon)^r (m - \varepsilon r a^r) a$ $(r+1)$-cliques of $H$ have exactly $r$ vertices in $A$ and one vertex in $\cup_{i \in [t]} B_i$, completing the proof. $\square$

The following consequence of Ramsey's theorem has been proved by Erdős [5].

**Lemma 3** *Given integers $p \geq 2$, $r \geq 2$, there exist a $c_{p,r} > 0$ such that if $G$ is a $K_{p+1}$-free graph of order $n$ and $n \geq r(K_{p+1}, K_r)$ then $G$ contains at least $c_{p,r} n^r$ independent $r$-sets.*

We need another result related to the regularity lemma of Szemerédi, the so-called Key Lemma (e.g., see [7], Theorem 2.1). We shall use the following simplified version of the Key Lemma.

**Theorem 5** *Suppose $0 < \varepsilon < d < 1$ and let $m$ be a positive integer. Let $G$ be a graph of order $(p+1)m$ and let $V(G) = V_1 \cup \cdots \cup V_{p+1}$ be a partition of $V(G)$ into $p+1$ sets of cardinality $m$ so that each of the pairs $(V_i, V_j)$ is $\varepsilon$-regular and has density at least $d$. If $\varepsilon \leq (d - \varepsilon)^p / (p+2)$ then $K_{p+1} \subset G$.*

**Proof of Theorem 3** Our proof is straightforward but rather rich in technical details, so we shall briefly outline it first. For some properly selected $\varepsilon$, applying SRL, we partition all but $\varepsilon n$ vertices of $G$ in $k$ sets $V_1, \ldots, V_k$ of equal cardinality such that almost all pairs $(V_i, V_j)$ are $\varepsilon$-regular. We may assume that the number of dense $\varepsilon$-regular pairs $(V_i, V_j)$ is no more than $\frac{p-1}{2p} k^2$, since otherwise, from Theorem 5 and Turán's theorem, $G$ will contain a $K_{p+1}$.

Therefore, there are at least $(1/2p+o(1))k^2$ sparse $\varepsilon$-regular pairs $(V_i, V_j)$. From Lemma 3 it follows that the number of independent $r$-sets in any of the sets $V_1, \ldots, V_k$ is $\Theta(n^r)$. Consider the size of the $r$-book in $\overline{G}$ having for its base the average independent $r$-set in $V_i$. For every sparse $\varepsilon$-regular pair $(V_i, V_j)$ almost every vertex in $V_j$ is a page of such a book. Also each $\varepsilon$-regular pair $(V_i, V_j)$ whose density is not very close to 1 contributes substantially many additional pages to such books. Precise estimates show that either $bs^{(r)}(\overline{G}) > n/p$ or else the number of all $\varepsilon$-regular pairs $(V_i, V_j)$ with density close to 1 is $\left(\frac{p-1}{2p} + o(1)\right) k^2$. Thus the size of $G$ is $\left(\frac{p-1}{2p} + o(1)\right) n^2$ and therefore, according to Theorem 2, $G$ contains the required induced $p$-chromatic subgraph with the required minimum degree.

*Details of the proof.* Let $c(p)$ be as in Theorem 2 and $c_{p,r}$ be as in Lemma 3. Select

$$\delta = \min\left\{\frac{\xi^3}{32}, \frac{c(p)}{4}\right\}, \tag{9}$$

set

$$d = \min\left\{\left(\frac{\delta}{2}\right)^{r+1}\left(\frac{r}{c_{p,r}} + 2r + 1 + 2p\right)^{-1}, \frac{p\delta}{1+p\delta}\left(\frac{r}{c_{p,r}} + 2r + 1\right)^{-1}\right\}, \tag{10}$$

and let

$$\varepsilon = \min\left\{\delta, \frac{d^p}{2\,(p+1)}\right\}. \tag{11}$$

These definitions are justified at the later stages of the proof. Since $c_{p,r} < r!$ we easily see that $0 < 2\varepsilon < d < \delta < 1$. Hence, Bernoulli's inequality implies

$$(d-\varepsilon)^p \geq d^p - p\varepsilon d^{p-1} > d^p - p\varepsilon = 2(p+1)\varepsilon - p\varepsilon = (p+2)\varepsilon. \tag{12}$$

Applying SRL we find a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_k$ so that $|V_0| < \varepsilon n$, $|V_1| = \cdots = |V_k|$ and all but $\varepsilon k^2$ pairs $(V_i, V_j)$ are $\varepsilon$-regular. Without loss of generality we may assume $|V_i| > r(K_{p+1}, K_r)$ and $k > 1/\varepsilon$. Consider the graphs $H_{\mathrm{irr}}$, $H_{\mathrm{lo}}$, $H_{\mathrm{mid}}$ and $H_{\mathrm{hi}}$ defined on the vertex set $[k]$ as follows:

(i) $(i,j) \in E(H_{\mathrm{irr}})$ iff the pair $(V_i, V_j)$ is not $\varepsilon$-regular,

(ii) $(i,j) \in E(H_{\mathrm{lo}})$ iff the pair $(V_i, V_j)$ is $\varepsilon$-regular and

$$d(V_i, V_j) \leq d,$$

(iii) $(i, j) \in E(H_{\text{mid}})$ iff the pair $(V_i, V_j)$ is $\varepsilon$-regular and

$$d < d(V_i, V_j) \le 1 - \delta,$$

(iv) $(i, j) \in E(H_{\text{hi}})$ iff the pair $(V_i, V_j)$ is $\varepsilon$-regular and

$$d(V_i, V_j) > 1 - \delta.$$

Clearly, no two of these graphs have edges in common; thus

$$e(H_{\text{irr}}) + e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) = \binom{k}{2}.$$

Hence, from $d > 2\varepsilon$ and $k > 1/\varepsilon$, we see that

$$\begin{aligned} e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) &\ge \binom{k}{2} - \varepsilon k^2 = \frac{k^2}{2} - \frac{k}{2} - \varepsilon k^2 \\ &\ge \frac{k^2}{2} - \varepsilon k^2 - \varepsilon k^2 > \left(\frac{1}{2} - d\right) k^2. \end{aligned} \tag{13}$$

Since $G$ is $K_{p+1}$-free, from (12), we have $\varepsilon \le (d - \varepsilon)^p / (p + 2)$; applying Theorem 5, we conclude that the graph $H_{\text{mid}} \cup H_{\text{hi}}$ is $K_{p+1}$-free. Therefore, from Turán's theorem,

$$e(H_{\text{mid}}) + e(H_{\text{hi}}) \le \left(\frac{p-1}{2p}\right) k^2,$$

and from inequality (13) we deduce

$$e(H_{\text{lo}}) > \left(\frac{1}{2p} - d\right) k^2. \tag{14}$$

Next we shall bound $bs^{(r)}(\overline{G})$ from below. To achieve this we shall count the independent $(r+1)$-sets having exactly $r$ vertices in some $V_i$ and one vertex outside $V_i$. Fix $i \in [k]$ and let $m$ be the number of independent $r$-sets in $V_i$. Observe that Lemma 3 implies $m \ge c_{p,r}|V_i|^r$.

Set $L = N_{H_{\text{lo}}}(i)$ and apply Lemma 2 with $A = V_i$, $B_j = V_j$, for all $j \in L$, and

$$H = \overline{G}\left[A \cup \left(\bigcup_{j \in L} B_j\right)\right].$$

Since, for every $j \in L$, the pair $(V_i, V_j)$ is $\varepsilon$-regular and

$$e_H(V_i, V_j) \geq (1-d)|V_i||V_j|,$$

we conclude that there are at least

$$d_{H_{\text{lo}}}(i)|V_i|(m - \varepsilon r|V_i|^r)(1-d-\varepsilon)^r$$

independent $(r+1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex in $\cup_{j\in L} B_j$.

Set now $M = N_{H_{\text{mid}}}(i)$, and apply Lemma 2 with $A = V_i$, $B_j = V_j$ for all $j \in M$ and

$$H = \overline{G}\left[A \cup \left(\bigcup_{j\in M} B_j\right)\right].$$

Since, for every $j \in M$, the pair $(V_i, V_j)$ is $\varepsilon$-regular and

$$e_H(V_i, V_j) \geq \delta|V_i||V_j|,$$

we conclude that there are at least

$$d_{H_{\text{mid}}}(i)|V_i|\left(m - \varepsilon r|V_i|^r\right)(\delta-\varepsilon)^r$$

independent $(r+1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex in $\cup_{j\in L} B_j$. Since

$$\left(\bigcup_{j\in L} B_j\right)\bigcap\left(\bigcup_{j\in M} B_j\right) = \varnothing,$$

there are at least

$$d_{H_{\text{lo}}}(i)|V_i|\left(m - \varepsilon r|V_i|^r\right)(1-d-\varepsilon)^r + d_{H_{\text{mid}}}(i)|V_i|\left(m - \varepsilon r|V_i|^r\right)(\delta-\varepsilon)^r$$

independent $(r+1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex outside $V_i$. Thus, taking the average over all $m$ independent $r$-sets in $V_i$, we conclude

$$\begin{aligned} bs^{(r)}\left(\overline{G}\right) &\geq |V_i|\left(1 - \frac{\varepsilon r}{c_{p,r}}\right)\left(d_{H_{\text{lo}}}(i)\,(1-d-\varepsilon)^r + d_{H_{\text{mid}}}(i)\,(\delta-\varepsilon)^r\right) \\ &\geq n\left(\frac{1-\varepsilon}{k}\right)\left(1 - \frac{\varepsilon r}{c_{p,r}}\right)\left(d_{H_{\text{lo}}}(i)\,(1-d-\varepsilon)^r + d_{H_{\text{mid}}}(i)\,(\delta-\varepsilon)^r\right). \end{aligned}$$

Summing this inequality for all $i=1,\ldots,k$ we obtain

$$\begin{aligned}
\frac{bs^{(r)}\left(\overline{G}\right)}{n} &\geq (1-\varepsilon)\left(1-\frac{\varepsilon r}{c_{p,r}}\right)\left(\frac{2e\left(H_{\mathrm{lo}}\right)}{k^2}(1-d-\varepsilon)^r+\frac{2e\left(H_{\mathrm{mid}}\right)}{k^2}(\delta-\varepsilon)^r\right)\\
&> \left(1-\left(\frac{r}{c_{p,r}}+1\right)\varepsilon\right)\left(\frac{2e\left(H_{\mathrm{lo}}\right)}{k^2}(1-r(d+\varepsilon))+\frac{2e\left(H_{\mathrm{mid}}\right)}{k^2}(\delta-\varepsilon)^r\right)\\
&> \left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)\left(\frac{2e\left(H_{\mathrm{lo}}\right)}{k^2}(1-2rd)+\frac{2e\left(H_{\mathrm{mid}}\right)}{k^2}\left(\frac{\delta}{2}\right)^r\right)\\
&> \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)\frac{2e\left(H_{\mathrm{lo}}\right)}{k^2}+\left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)\left(\frac{\delta}{2}\right)^r\frac{2e\left(H_{\mathrm{mid}}\right)}{k^2}.
\end{aligned} \tag{15}$$

Assume the assertion of the theorem false and suppose

$$bs^{(r)}\left(\overline{G}\right)\leq\frac{n}{p}. \tag{16}$$

We shall prove that this assumption implies

$$e\left(H_{\mathrm{lo}}\right)<\left(\frac{1}{2p}+\frac{\delta}{2}\right)k^2, \tag{17}$$

$$e\left(H_{\mathrm{mid}}\right)<\delta k^2. \tag{18}$$

Disregarding the term $e\left(H_{\mathrm{mid}}\right)$ in (15), in view of (16) and (10), we have

$$\begin{aligned}
e\left(H_{\mathrm{lo}}\right) &< \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)^{-1}\frac{bs^{(r)}\left(\overline{G}\right)}{2n}k^2\\
&\leq \left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)^{-1}\frac{k^2}{2p}\\
&\leq \left(1-\frac{p\delta}{1+p\delta}\right)^{-1}\frac{k^2}{2p}=\left(\frac{1}{2p}+\frac{\delta}{2}\right)k^2,
\end{aligned}$$

and inequality (17) is proved.

Furthermore, observe that equality (10) implies

$$\left(\frac{r}{c_{p,r}}+1\right)d<\left(\frac{r}{c_{p,r}}+2r+1\right)d\leq\frac{p\delta}{1+p\delta}\leq p\delta<\frac{1}{2},$$

and consequently,

$$\left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)>\frac{1}{2}.$$

Hence, from (15), taking into account (16) and (14), we find that

$$\begin{aligned}\frac{e\left(H_{\text{mid}}\right)}{2}\left(\frac{\delta}{2}\right)^{r} &< e\left(H_{\text{mid}}\right)\left(\frac{\delta}{2}\right)^{r}\left(1-\left(\frac{r}{c_{p,r}}+1\right)d\right)\\
&\leq \frac{bs^{(r)}\left(\overline{G}\right)k^{2}}{2n}-\left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)e\left(H_{\text{lo}}\right)\\
&< \left(\frac{1}{2p}-\left(1-\left(\frac{r}{c_{p,r}}+2r+1\right)d\right)\left(\frac{1}{2p}-d\right)\right)k^{2}\\
&= \left(1+\left(\frac{r}{c_{p,r}}+2r+1\right)\left(\frac{1}{2p}-d\right)\right)dk^{2}\\
&< \frac{1}{2p}\left(\frac{r}{c_{p,r}}+2r+1+2p\right)dk^{2}<\left(\frac{\delta}{2}\right)^{r+1}k^{2}.\end{aligned}$$

Therefore, inequality (18) holds also.

Furtermore, inequality (13), together with (17) and (18), implies

$$e\left(H_{\text{hi}}\right)>\left(\frac{1}{2}-d\right)k^{2}-\left(\frac{1}{2p}+\frac{\delta}{2}\right)k^{2}-\delta k^{2}=\left(\frac{p-1}{2p}-\frac{5\delta}{2}\right)k^{2},$$

and consequently, from the definition of $H_{\text{hi}}$, we obtain

$$\begin{aligned}e\left(G\right)&\geq e\left(H_{\text{hi}}\right)\left(\frac{\left(1-\varepsilon\right)n}{k}\right)^{2}\left(1-\delta\right)>\left(\frac{p-1}{2p}-\frac{5\delta}{2}\right)\left(1-2\varepsilon\right)\left(1-\delta\right)n^{2}\\
&=\frac{p-1}{2p}\left(1-\frac{5p\delta}{p-1}\right)\left(1-2\varepsilon\right)\left(1-\delta\right)n^{2}>\\
&>\frac{p-1}{2p}\left(1-\left(\frac{5p}{p-1}+3\right)\delta\right)n^{2}>\left(\frac{p-1}{2p}-4\delta\right)n^{2}.\end{aligned}$$

Hence, by (9), applying Theorem 2, it follows that $G$ contains an induced $p$-chromatic graph with the required properties. $\square$

Following the basic idea of the proof of Theorem 3 but applying the complete Key Lemma instead of Theorem 5, we obtain a more general result, whose proof, however, is considerably

easier than the proof of Theorem 3.

**Theorem 6** *Suppose $H$ is a fixed $(p+1)$-chromatic graph. For every $H$-free graph $G$ of order $n$,*

$$bs^{(r)}\left(\overline{G}\right) > \left(\frac{1}{p} + o\left(1\right)\right) n.$$

$\square$

Note that the graph $K_p\left(q+r-1\right)$ is $p$-chromatic and its complement has no $B_q^{(r)}$, so for every $(p+1)$-chromatic graph $H$ and every $r, q$ we have

$$r\left(H, B_q^{(r)}\right) \geq p\left(q+r-1\right)+1.$$

Hence, from Theorem 6, we immediately obtain the following theorem.

**Theorem 7** *For every fixed $(p+1)$-chromatic graph $H$ and fixed integer $r > 1$,*

$$r\left(H, B_q^{(r)}\right) = pq + o\left(q\right).$$

$\square$

Note that it is not possible to avoid the $o\left(q\right)$ term in Theorem 7 without additional stipulations about $H$, since, as Faudree, Rousseau and Sheehan have shown in [6], the inequality

$$r\left(C_4, B_q^{(2)}\right) \geq q + 2\sqrt{q}$$

holds for infinitely many values of $q$. However, when $H = K_{p+1}$ and $q$ is large we can prove a precise result.

# 4 Ramsey numbers $r\left(K_p, B_q^{(r)}\right)$ for large $q$

In this section we determine $r\left(K_p, B_q^{(r)}\right)$ for fixed $p \geq 3$, $r \geq 2$ and large $q$.

**Theorem 8** *For fixed $p \geq 2$ and $r \geq 2$, $r(K_{p+1}, B_q^{(r)}) = p(q+r-1)+1$ for all sufficiently large $q$.*

**Proof** Since $K_p(q+r-1)$ contains no $K_{p+1}$ and its complement contains no $B_q^{(r)}$, we have

$$r(K_{p+1}, B_q^{(r)}) \geq p(q+r-1)+1.$$

Let $G$ be a $K_{p+1}$-free graph of order $n = p(q+r-1)+1$. Since $n/p > q$, either we're done or else $G$ contains an induced $p$-chromatic subgraph $G_1$ of order $pq + o(q)$ with minimum degree

$$\delta(G_1) \geq \left(1 - \frac{1}{p} + o(1)\right) n.$$

Using this bound on $\delta(G_1)$ we can easily prove by induction on $p$ that $G_1$ contains a copy of $K_p(r)$. Fix a copy of $K_p(r)$ in $G_1$ and let $A_1, A_2, \ldots, A_p$ be its vertex classes. Let $A = A_1 \cup \cdots \cup A_p$ and $B = V(G) \setminus A$. If some vertex $i \in B$ is adjacent to at least one vertex in each of the parts $A_1, A_2, \ldots, A_p$ then $G$ contains a $K_{p+1}$. Otherwise for each vertex $u \in B$ there is at least one $v$ so that $u$ is adjacent in $\overline{G}$ to all members of $A_v$. It follows by the pigeonhole principle that $bs^{(r)}(\overline{G}) = s$ where

$$s \geq \left\lceil \frac{n - p(r-1)}{p} \right\rceil = \left\lceil q - 1 + \frac{1}{p} \right\rceil = q,$$

and we really are done. $\square$

The proof using the regularity lemma that $r(K_{p+1}, B_q^{(r)}) = p\,(q+r-1)+1$ if $q$ is sufficiently large does indeed require that $q$ increase quite rapidly as a function of the parameters $p$ and $r$. This raises the question of what growth rate is actually required. The following simple calculation shows that polynomial growth in $p$ is not sufficient.

**Theorem 9** *For arbitrary fixed $k$ and $r$,*

$$\frac{r(K_m, B_{m^k}^{(r)})}{m^{k+r-1}} \to \infty$$

*as $m \to \infty$.*

**Proof** We shall prove that $r(K_m, B_{m^k}^{(r)}) > cm^{k+r}/(\log m)^r$ for all sufficiently large $m$. Let $N = \lfloor cm^{k+r}/(\log m)^r \rfloor$ where $c$ is to be chosen, and set $p = (C/m)\log m$ where $C =$

$2(k+r-1)$. Let $G$ be the random graph $G = G(N, 1-p)$. The probability that $K_m \subset G$

$$\mathbb{P}(K_m \subset G) \leq \binom{N}{m}(1-p)^{\binom{m}{2}} \leq \binom{N}{m} e^{-pm(m-1)/2} < \left(\frac{Ne}{m}\right)^m e^{pm/2} m^{-(k+r-1)m}$$

$$= \left(\frac{Ne^{1+p/2}m^{-(k+r-1)}}{m}\right)^m = o(1), \quad m \to \infty.$$

To bound the probability that $B^{(r)}_{m^k} \subset \overline{G}$, we use the following simple consequence of Chernoff's inequality [4]: if $X = X_1 + X_2 + \cdots + X_n$ where independently each $X_i = 1$ with probability $\mathfrak{p}$ and $X_i = 0$ with probability $1 - \mathfrak{p}$ then

$$\mathbb{P}(X \geq M) \leq \left(\frac{n\mathfrak{p}e}{M}\right)^M$$

for any $M \geq n\mathfrak{p}$. Thus we find

$$\mathbb{P}(B^{(r)}_{m^k} \subset \overline{G}) \leq \binom{N}{r} p^{r(r-1)/2} \left(\frac{(N-r)p^r e}{m^k}\right)^{m^k}.$$

Since the product of the first two factors has polynomial growth in $m$, to have $\mathbb{P}(B^{(r)}_{m^k}) = o(1)$ when $m \to \infty$, it suffices to take $c = 1/(3C^r)$, so that

$$\frac{(N-r)p^r e}{m^k} \leq \frac{(c\, m^{k+r}/(\log m)^r)((C/m)\log m)^r e}{m^k} = \frac{e}{3},$$

making the last factor approach 0 exponentially. $\square$

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